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The Geometrycal Interpretation of the Relations between the Laspeyres, Paasche, Fisher and Drobisch Indexes and a New Presentation of the Bortkiewicz Relation

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Abstract. The indexes numbers's rol is to measure and to interpret the average variation in time, space or in relation with other systems of reference, in relative sizes, of the levels for the variables who belong to the statistical units from a collectivity. The indexes achieve a statistical analysis of the movement and of the change for these phenomenons. Therefore, the indexes are extremely importants in to underline the dynamic life which surrounds us.

Keywords: indexes numbers; geometrycal interpretation; the Laspeyres index, the Paasche index, the Fisher index, the Drobisch index, the Bortkiewicz relation.

JEL Code: C1, C12, C2

1. Introduction

On the measure what the time flowes irreversible, we observe the existence of one real circuit traversed by a continuous wave of the changes, as the factors's actions effect, both in nature, in universe, and in any domain of the activity, transformations who influence the phenomenons's values who participate in the governation of the dynamic life.

So, through the agency of one certain the statistical indicatories's category, who it's represented by the statistical indexes, we can calculate the phenomenons's relative changes in time, space or in relation with other system of reference.

The indexes numbers award of the statistical language a new size and they express with a choice accuracy the changes from any domain of activity. In this sense, the Laspeyres indexes, the Paasche indexes, the Fisher indexes and the Drobisch indexes have an important rol in the universe of the indexes numbers. They put in evidence the progress or the regress from any domain examined, inclusively in the situation when the changes can not be grasp straightly.

2. Content

I'll introduce a **theoretical contribution** *with a view to reflect the geometrycal interpretation of the relations between the Laspeyres, Paasche, Fisher and Drobisch indexes*, concerning the case of the equality among the values of these indexes, as well as the interpretations in the cases of the inequality from the values of the same indexes.

Let us consider one circle, where in him interior there is a rectangular triangle ABC. In these case the hypotenuse of the triangle ABC dovetailes with the diameter of the circle.

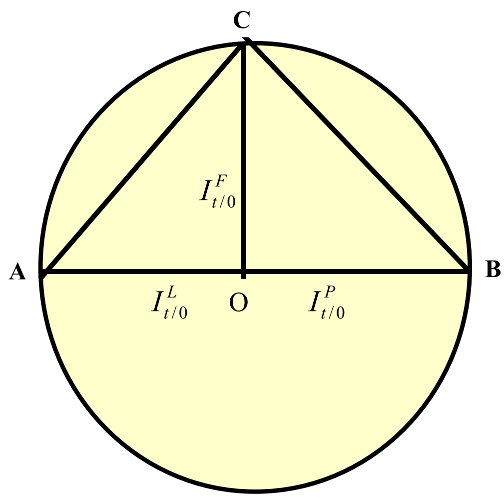


Figure 1. The geometrical interpretation of the equality between the Laspeyres, Paasche, Fisher and Drobisch indexes

We formulate the first assumption H_0: *therefore we suppose equals the values of the Laspeyres, Paasche, Fisher and Drobisch indexes, namely:*

$$I^L_{t/0} = I^P_{t/0} = I^F_{t/0} = I^D_{t/0}. \quad (1)$$

We'll use the next notations:

$$AO = I^L_{t/0} = r \quad (2)$$

and

$$BO = I^P_{t/0} = r \quad (3)$$

where r = the radius of the circle.

We'll apply the theorem of the height in the rectangular triangle ABC and we'll obtain:

$$CO^2 = AO \cdot BO \quad (4)$$

Therefore,

$$CO = \sqrt{AO \cdot BO} = \sqrt{I^L_{t/0} \cdot I^P_{t/0}} = \sqrt{r \cdot r} = \sqrt{r^2} = r \quad (5)$$

Also,

$$I^F_{t/0} = \sqrt{I^L_{t/0} \cdot I^P_{t/0}} \quad (6)$$

So, on the base of the relations (5) and (6), it results:

$$I^F_{t/0} = r \quad (7)$$

Also,

$$I^D_{t/0} = \frac{I^L_{t/0} + I^P_{t/0}}{2} = \frac{AO + BO}{2} = \frac{r + r}{2} = \frac{2r}{2} = r \quad (8)$$

In conclusion,

$$I^L_{t/0} = I^P_{t/0} = I^F_{t/0} = I^D_{t/0} = r \quad (9)$$

So, the relation (9) expresses the checking of the assumption H_0. Therefore, the single modality when the values of the Laspeyres, Paasche, Fisher ans Drobisch indexes are equals, is viable in the conditions when the indexes represent the rise of the circle, while the size of the Fisher index will be the perpendicular haggarded on the hypotenuse of the rectangular triangle which is the diameter of the circle when the respectively triangle is writes down, in accordance with the figure no. 1.

In continuation, we'll formulate the second assumption $H_0^{'}$: *therefore we suppose the existence of the next inequality:* $I_{t/0}^{L} > I_{t/0}^{D} > I_{t/0}^{F} > I_{t/0}^{P}$, *when the value of the Paache index is less face to the value of the Laspeyres,* $I_{t/0}^{P} < I_{t/0}^{L}$.

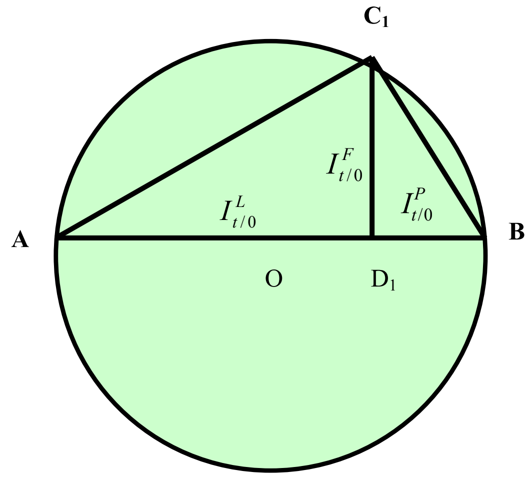


Figure 2. The geometrical interpretation of the inequality $I_{t/0}^{L} > I_{t/0}^{D} > I_{t/0}^{F} > I_{t/0}^{P}$, in the conditions when $I_{t/0}^{P} < I_{t/0}^{L}$.

So, we'll note in the figure no. 2:

$$AD_1 = I_{t/0}^{L} \tag{10}$$

and

$$BD_1 = I_{t/0}^{P} \tag{11}$$

Thus, it's expresses that we operate the checking of the assumption $H_o^{'}$, $I_{t/0}^{P} < I_{t/0}^{L}$.

Also, if we apply the theorem of the height in the rectangular ABC, we'll obtain:

$$C_1D_1 = \sqrt{AD_1 \cdot BD_1} = \sqrt{I_{t/0}^{L} \cdot I_{t/0}^{P}} \tag{12}$$

Therefore, the value of the index Fisher is equals with the lenght of the side C_1D_1:

$$I_{t/0}^{F} = \sqrt{I_{t/0}^{L} \cdot I_{t/0}^{P}} = C_1D_1 \tag{13}$$

Also, the size of the Drobisch index will be equal with the rise of the circle:

$$I_{t/0}^{D} = \frac{I_{t/0}^{L} + I_{t/0}^{P}}{2} = \frac{AD_1 + BD_1}{2} = \frac{AB}{2} = \frac{2r}{2} = r \tag{14}$$

Therefore, if we'll observe the figure no. 2 and we'll demonstrate in the situation when $I_{t/0}^{P} < I_{t/0}^{L}$, we can estimate that anywhere the perpendicular C_1D_1 falls on the segment OB, we'll obtain:

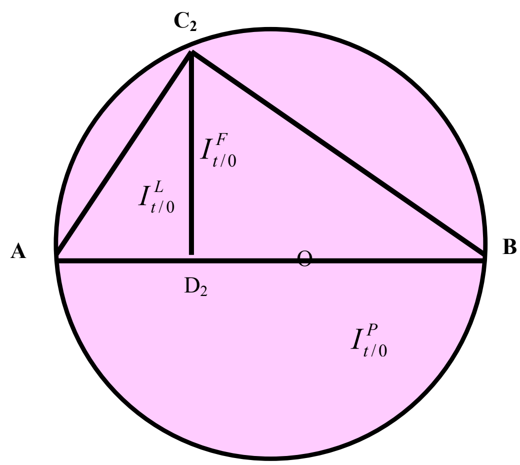
$$I_{t/0}^{L} > I_{t/0}^{D} > I_{t/0}^{F} > I_{t/0}^{P} \tag{15}$$

because,

$$AD_1 > r > C_1D_1 > BD_1 \tag{16}$$

So, this relation reflects the checking of the second assumption $H_0^{'}$.

Concomitantly, we formulate the third assumption $H_0^{"}$: therefore we suppose the existence of the next inequality: $I_{t/0}^{L} < I_{t/0}^{F} < I_{t/0}^{D} < I_{t/0}^{P}$, when the size of the Paache index is more big face to the value of the Laspeyres index, $I_{t/0}^{P} > I_{t/0}^{L}$.



The figure number 3 - The geometrical interpretation of the inequality $I_{t/0}^{L} < I_{t/0}^{F} < I_{t/0}^{D} < I_{t/0}^{P}$, in the condition when $I_{t/0}^{P} > I_{t/0}^{L}$.

Analogous, we'll use the next notations in the figure no.3:

$$AD_2 = I_{t/0}^{L} \tag{17}$$

and

$$BD_2 = I_{t/0}^{P} \tag{18}$$

Thus, we achieve the checking of the assumption $H_0^{'}$, in the condition when $I_{t/0}^{P} > I_{t/0}^{L}$.

We know in accordance with the theorem of the height, that the length of the perpendicular who was drawed from the tip of the angle of the rectangular triangle ABC_2, on his hypotenuse, it will calculate on the base of the formula:

$$C_2D_2 = \sqrt{AD_2 \cdot BD_2} = \sqrt{I_{t/0}^{L} \cdot I_{t/0}^{P}} \tag{19}$$

Therfore, the size of the Fisher index will be equals with the length of the side C_2D_2:

$$I_{t/0}^{F} = \sqrt{I_{t/0}^{L} \cdot I_{t/0}^{P}} = C_2D_2 \tag{20}$$

Also, the value of the Drobisch index will be equals with the rise of the circle:

$$I_{t/0}^{D} = \frac{I_{t/0}^{L} + I_{t/0}^{P}}{2} = \frac{AD_2 + BD_2}{2} = \frac{AB}{2} = \frac{2r}{2} = r \tag{21}$$

In conclusion, if we observe the figure number 3, we can specify that anywhere it's the extreme of the perpendicular C_2D_2 on the segment AO, we'll have:

$$I_{t/0}^{L} < I_{t/0}^{F} < I_{t/0}^{D} < I_{t/0}^{P} \tag{22}$$

as,

$$AD_2 < C_2D_2 < r < BD_2 \tag{23}$$

Therefore, it's express the checking of the third assumption $H_0^{''}$, in the conditions when $I_{t/0}^{P} > I_{t/0}^{L}$.

If we synthesize, we observe on the base of the figures number 2 and 3. and of the relations (15) şi (22), that indifferent where there is the projection of the perpendicular on the diameter AB of the circle, the value of the Fisher index will be less face to the value of the Drobisch index, in the conditions when the both situations are carryed out, namely when the size of the Paasche index is less face to the value of the Laspeyres index, respectively when it's manifest the superiority of the value who belongs to the Paasche index in relation with the value of the Laspeyres index.

I'll present in continuation another **one's own theoretical contribution** who reflects *a new modality for to express the Bortkiewicz relation.*

Thus, we'll formulate the assumption H_0: who suppose that the difference from the Paasche index and Laspeyres index who are builted for to express the dynamic of the prices at the level of the products $i = \overline{1,n}$, will be equals with the double negative value, respectively wuth the double positive value, of the radical who represents the deviation from the square of the symple arithmetical average of the Laspeyres and Paasche prices indexes for the respective products and the square of the symple geometrycal average of the Laspeyres and Paasche indexes who presents the variation of the prices on the same products $i = \overline{1,n}$ taked in observation.

For to check up this assumption, we'll start from to calculate the square of the ratio between the Fisher prices indexes and the Drobisch prices index of the products $i = \overline{1,n}$, from wich we'll substract the unit, such as we can observe in the next relation:

$$\left[\frac{I_{t/0}^{F(p)}}{I_{t/0}^{D(p)}}\right]^2 - 1 = \frac{[I_{t/0}^{F(p)}]^2 - [I_{t/0}^{D(p)}]^2}{[I_{t/0}^{D(p)}]^2} = \frac{\mathrm{I}_{t/0}^{L(p)} \cdot \mathrm{I}_{t/0}^{P(p)} - \frac{[\mathrm{I}_{t/0}^{L(p)} + \mathrm{I}_{t/0}^{P(p)}]^2}{4}}{\frac{[\mathrm{I}_{t/0}^{L(p)} + \mathrm{I}_{t/0}^{P(p)}]^2}{4}} =$$

$$= \frac{4 \cdot I_{t/0}^{L(p)} \cdot I_{t/0}^{P(p)} - [I_{t/0}^{L(p)}]^2 - 2 \cdot I_{t/0}^{L(p)} \cdot I_{t/0}^{P(p)} - [I_{t/0}^{P(p)}]^2}{[I_{t/0}^{L(p)} + I_{t/0}^{P(p)}]^2} = -\frac{[\mathrm{I}_{t/0}^{L(p)} - \mathrm{I}_{t/0}^{P(p)}]^2}{[\mathrm{I}_{t/0}^{L(p)} + \mathrm{I}_{t/0}^{P(p)}]^2} \qquad (24)$$

Therefore,

$$[I_{t/0}^{L(p)} - I_{t/0}^{P(p)}]^2 = [I_{t/0}^{L(p)} + I_{t/0}^{P(p)}]^2 \cdot \left[1 - \left(\frac{I_{t/0}^{F(p)}}{I_{t/0}^{D(p)}}\right)^2\right]$$

(25)

Thus,

$$I_{t/0}^{L(p)} - I_{t/0}^{P(p)} = \pm \frac{I_{t/0}^{L(p)} + I_{t/0}^{P(p)}}{I_{t/0}^{D(p)}} \cdot \sqrt{[I_{t/0}^{D(p)}]^2 - [I_{t/0}^{F(p)}]^2} =$$

$$= \pm \frac{I_{t/0}^{L(p)} + I_{t/0}^{P(p)}}{\frac{I_{t/0}^{L(p)} + I_{t/0}^{P(p)}}{2}} \cdot \sqrt{[I_{t/0}^{D(p)}]^2 - [I_{t/0}^{F(p)}]^2} = \pm 2 \cdot \sqrt{[\mathrm{I}_{t/0}^{D(p)}]^2 - [\mathrm{I}_{t/0}^{F(p)}]^2} \qquad (26)$$

Therefore,

$$I_{t/0}^{L(p)} - I_{t/0}^{P(p)} = \pm 2 \cdot \sqrt{[I_{t/0}^{D(p)}]^2 - [I_{t/0}^{F(p)}]^2} \qquad (27)$$

In conclusion,

$$I_{t/0}^{P(p)} - I_{t/0}^{L(p)} = \mp 2 \cdot \sqrt{[I_{t/0}^{D(p)}]^2 - [I_{t/0}^{F(p)}]^2} \qquad (28)$$

where:

$I_{t/0}^{L(p)}; I_{t/0}^{P(p)}; I_{t/0}^{D(p)}; I_{t/0}^{F(p)}$ = the Laspeyres, Paasche, Drobisch and Fisher prices indexes of the products $i = \overline{1,n}$.

So, we can observe that the assumption H_0 it checks up. Also, in accordance with the relations (9), (15) and (22), we know that the value of the Fisher index who reflects the average variation of the prices for the products $i = \overline{1,n}$, is less or equal face to the size of the Drobisch index who expresses the average dynamic of the prices for the same products:

$$I_{t/0}^{F(p)} \le I_{t/0}^{D(p)} \quad (29)$$

while the effect will be the achievement of the inequalities:

$$[I_{t/0}^{F(p)}]^2 \le [I_{t/0}^{D(p)}]^2 \quad (30)$$

respectivelly,

$$[I_{t/0}^{D(p)}]^2 - [I_{t/0}^{F(p)}]^2 \ge 0 \quad (31)$$

Therefore, the next situations appear:

a) if $I_{t/0}^{F(p)} = I_{t/0}^{D(p)}$, then $[I_{t/0}^{D(p)}]^2 - [I_{t/0}^{F(p)}]^2 = 0$, and if we appeal to the relation (28), vom avea $I_{t/0}^{L(p)} = I_{t/0}^{P(p)}$. So, if we observe the demonstrations achieved in the relations (1)–(7), we'll obtain the equality $I_{t/0}^{L(p)} = I_{t/0}^{P(p)} = I_{t/0}^{F(p)} = I_{t/0}^{D(p)}$ from the relation (8), who is viable in the conditions when the values of the Laspeyres, Paasche, Fisher and Drobisch prices indexes of the products $i = \overline{1,n}$, are equal with the rise of the circle, situation presented in the figure number 1.

b) if $I_{t/0}^{F(p)} < I_{t/0}^{D(p)}$, respectivelly $[I_{t/0}^{D(p)}]^2 - [I_{t/0}^{F(p)}]^2 > 0$, then:

- if the changes of reverse sense between the individual indexes of the prices and the individual indexes of the quantities prevail at the level of the each product who was researched, then we'll obtain $I_{t/0}^{P(p)} < I_{t/0}^{L(p)}$, that is $I_{t/0}^{P(p)} - I_{t/0}^{L(p)} < 0$, situation who reflects the utilization of the next relation in calculation:

$$I_{t/0}^{P(p)} - I_{t/0}^{L(p)} = -2 \cdot \sqrt{[I_{t/0}^{D(p)}]^2 - [I_{t/0}^{F(p)}]^2} \quad (32)$$

- if the changes of same sense between the individual indexes of the prices and the individual indexes of the quantities dominate for each product who was taked in observation, then we'll have $I_{t/0}^{P(p)} > I_{t/0}^{L(p)}$, namely $I_{t/0}^{P(p)} - I_{t/0}^{L(p)} > 0$, case who has as effect the utilization of the next relation in our calculation:

$$I_{t/0}^{P(p)} - I_{t/0}^{L(p)} = +2 \cdot \sqrt{[I_{t/0}^{D(p)}]^2 - [I_{t/0}^{F(p)}]^2} \quad (33)$$

Analogous, we can obtain the difference from the Paasche indexes of the quantities for the products $i = \overline{1,n}$ and the Laspeyres indexes of the same quantities who are analysed, if we start from the calculation of the square of the ratio between the Fisher indexes and the Drobisch indexes who express the average level of the dynamic for the quantities of the products $i = \overline{1,n}$:

$$I_{t/0}^{P(q)} - I_{t/0}^{L(q)} = \mp 2 \cdot \sqrt{[I_{t/0}^{D(q)}]^2 - [I_{t/0}^{F(q)}]^2} \quad (34)$$

Also, the next cases will appear:

a) if $I_{t/0}^{F(q)} = I_{t/0}^{D(q)}$, then $[I_{t/0}^{D(q)}]^2 - [I_{t/0}^{F(q)}]^2 = 0$, and if we'll apply the relations (34) and (9), then we'll obtain $I_{t/0}^{L(q)} = I_{t/0}^{P(q)}$, respectivelly $I_{t/0}^{L(q)} = I_{t/0}^{P(q)} = \; = I_{t/0}^{F(q)} = I_{t/0}^{D(q)}$, equality who is really in the situation when the sizes separated of the Laspeyres, Paasche, Fisher and Drobisch indexes of the quantities for the products $i = \overline{1,n}$, are equals with the rise of the circle, in acordance with the figure number 1.

b) if $I_{t/0}^{F(q)} < I_{t/0}^{D(q)}$, respectivelly $[I_{t/0}^{D(q)}]^2 - [I_{t/0}^{F(q)}]^2 > 0$, and if we appeal to the relation $\frac{I_{t/0}^{P(q)}}{I_{t/0}^{L(q)}} = \frac{I_{t/0}^{P(p)}}{I_{t/0}^{L(p)}} = 1 + r_{i_{t/0}^{q_i}, i_{t/0}^{p_i}} \cdot v_{t/0}^{q_i} \cdot v_{t/0}^{p_i}$, then:

- if the changes of reverse sense between the individual indexes of the prices and the individual indexes of the quantities prevail at the level of the each product who was taked in study, then we'll have $I_{t/0}^{P(q)} < I_{t/0}^{L(q)}$, that is $I_{t/0}^{P(q)} - I_{t/0}^{L(q)} < 0$, situation who imposes the utilization of the relation:

$$I_{t/0}^{P(q)} - I_{t/0}^{L(q)} = -2 \cdot \sqrt{[I_{t/0}^{D(q)}]^2 - [I_{t/0}^{F(q)}]^2} \qquad (35)$$

- if there is a dominion of the changes of the same sense between the individual indexes of the prices and the individual indexes of the quantities for each product who is analysed, then we'll obtain $I_{t/0}^{P(q)} > I_{t/0}^{L(q)}$, namelly $I_{t/0}^{P(q)} - I_{t/0}^{L(q)} > 0$, case who contributes at the utilization of the next relation:

$$I_{t/0}^{P(q)} - I_{t/0}^{L(q)} = +2 \cdot \sqrt{[I_{t/0}^{D(q)}]^2 - [I_{t/0}^{F(q)}]^2} \qquad (36)$$

3. Conclusions

In synthesis, the relations (28) şi (34) reflect the priority in these formulas of the minus sign and then of the plus sign, in accordance with the opinions of the experts from statistics, namely in the practice we meet dense the inequalities $I_{t/0}^{P(p)} - I_{t/0}^{L(p)} < 0$ and $I_{t/0}^{P(q)} - I_{t/0}^{L(q)} < 0$. Therefore, often, the values of the Paasche indexes are less face to the values of the Laspeyres indexes. Also, the relations (28) şi (34) reflect the dependence who exist between the most important synthetic indexes: the Laspeyres, Paasche, Fisher and Drobisch indexes who represent the real stars from the life of the indexes.

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